Frequency-Dependent Microphone Matrices

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1 Introduction

Consider an array of three coincident microphones surrounded by four speakers. We wish to represent any sound as accurately as possible through the speaker matrix, such that the voltage function recorded is the same as the original voltage of the speakers. First we consider the case that the microphones are not frequency-dependent, implying the microphone pickup functions are solely dependent on the angle each respective microphone is pointed. We then consider the more realistic scenario that the microphone pickup functions are dependent on frequency.

Given a microphone directed at an angle \( \theta \) with pickup function \( \mu(\theta; \vartheta) \), and sound field function \( f(t, \theta) \), the voltage function of the microphone is given by the equation

\[
v(t, \vartheta) = \int_{-\pi}^{\pi} \mu(\theta; \vartheta) f(t, \theta) d\theta
\]

We wish to equally distribute the coincident microphones over the unit circle, such that \( \vartheta = 0, \frac{2\pi}{3}, \frac{4\pi}{3} \). Moreover, we place the four speakers at angles \( \phi, -\phi, \phi + \pi, -\phi - \pi \). We wish to find the angle \( \phi \) at which the microphone-speaker array best reproduces the original sound field function. Since we seek an angle \( \phi \) such that each speaker lies in the four quadrants of the xy-plane, we restrict \( \phi \in (0, \pi/2) \).

1.1 Assumptions

For ease of calculations, we assume both the microphone pickup function and sound field function are of degree one, such that their Fourier expansions can be represented

\[
\mu(\theta; \vartheta) = a_0 + a_1 \cos(\theta - \vartheta)
\]

\[
f(t, \theta) = A_0(t) + A_1(t) \cos \theta + B_1(t) \sin \theta
\]

Where \( a_0, a_1 \in \mathbb{C} \) are the Fourier coefficients for the pickup function and \( A_0, A_1, B_1 \in \mathcal{F}(\mathbb{R}) \) are the Fourier coefficients for the sound field function. Note that the microphone pickup function is a Fourier cosine series. This is due to the fact that the microphone pickup function is assumed to be an even function, such that \( \mu(\theta; \vartheta) = \mu(-\theta; \vartheta) \). Moreover, we assume the speakers are unidirectional such that the overall speaker sound function can be represented

\[
f_{\text{speaker}}(t, \theta) = s_1(t)\delta(\theta - \phi) + s_2(t)\delta(\theta + \phi) + s_3(t)\delta(\theta - \phi - \pi) + s_4(t)\delta(\theta + \phi + \pi)
\]

Where \( s_1, s_2, s_3, s_4 \) are the individual speaker sound field functions and \( \delta \) represents the Dirac measure. To reiterate, we wish to find the angle \( \phi \) such that

\[
v(t, \vartheta) = \int_{-\pi}^{\pi} f(t, \theta) \mu(\theta; \vartheta) d\theta = \int_{-\pi}^{\pi} f_{\text{speaker}}(t, \theta) \mu(\theta; \vartheta) d\theta = v_{\text{speaker}}(t, \vartheta)
\]

2 Solution to Frequency-Independent Case

First, we consider the voltage function of the microphones for the original sound field, \( f(t, \theta) \). Based on the assumptions above in (2) and (3), we can represent the voltage function as

\[
v(t, \vartheta) = \int_{-\pi}^{\pi} (A_0(t) + A_1(t) \cos \theta + B_1(t) \sin \theta)(a_0 + a_1 \cos(\theta - \vartheta))d\theta
\]

\[
= \int_{-\pi}^{\pi} a_0 A_0(t) + a_0 A_1(t) \cos \theta + a_0 B_1(t) \sin \theta + a_1 A_0(t) \cos(\theta - \vartheta) + a_1 A_1(t) \cos \theta \cos(\theta - \vartheta) + a_1 B_1(t) \sin \theta \cos(\theta - \vartheta)d\theta
\]

\[
= 2\pi a_0 A_0(t) + \pi a_1 \cos \vartheta A_1(t) + \pi a_1 \sin \vartheta B_1(t)
\]
We now consider the voltage function, \( v_{\text{speaker}} \), of the microphone after picking up the sound field emitted from the surrounding stereos. It follows from (4) that

\[
v_{\text{speaker}}(t, \overline{\theta}) = \int_{-\pi}^{\pi} (a_0 + a_1 \cos(\theta - \overline{\theta})) \left( s_1(t) \delta(\theta - \phi) + s_2(t) \delta(\theta + \phi) + s_3(t) \delta(\theta - \phi - \pi) + s_4(t) \delta(\theta + \phi + \pi) \right) d\theta
\]

\[
= a_0 \left( s_1(t) + s_2(t) + s_3(t) + s_4(t) \right) + a_1 \left( s_1(t) \cos(\phi - \overline{\theta}) + s_2(t) \cos(\phi + \theta) + s_3(t) \cos(\phi + \pi - \overline{\theta}) + s_4(t) \cos(\phi + \pi + \overline{\theta}) \right)
\]

As stated above, to reproduce the original sound field through the speakers \( v(t, \overline{\theta}) = v_{\text{speaker}}(t, \overline{\theta}) \) must hold. Thus, we group terms with similar coefficients, such that

\[
2\pi a_0 A_0(t) = a_0 \left( s_1(t) + s_2(t) + s_3(t) + s_4(t) \right)
\]

and

\[
\pi a_1 (A_1(t) \cos \overline{\theta} + B_1(t) \sin \overline{\theta}) = a_1 \left( s_1(t) \cos(\phi - \overline{\theta}) + s_2(t) \cos(\phi + \theta) + s_3(t) \cos(\phi + \pi - \overline{\theta}) + s_4(t) \cos(\phi + \pi + \overline{\theta}) \right)
\]

We can easily solve for \( A_0(t) \) in terms of \( s_1(t), s_2(t), s_3(t), s_4(t) \). However, in order to group similar terms for \( A_1(t) \) and \( B_1(t) \), we must use the sum/difference formulae for cosine. Thus, we get

\[
A_1(t) \cos \overline{\theta} + B_1(t) \sin \overline{\theta} = \frac{1}{\pi} \left( s_1(t)(\cos \phi \cos \overline{\theta} + \sin \phi \sin \overline{\theta}) + s_2(t)(\cos \phi \cos \overline{\theta} - \sin \phi \sin \overline{\theta}) 
+ s_3(t)(\cos(\phi + \pi) \cos \overline{\theta} + \sin(\phi + \pi) \sin \overline{\theta}) + s_4(t)(\cos(\phi + \pi) \cos \overline{\theta} - \sin(\phi + \pi) \sin \overline{\theta}) \right)
\]

\[
= \frac{1}{\pi} \cos \overline{\theta} \left( s_1(t) \cos \phi + s_2(t) \cos \phi + s_3(t) \cos(\phi + \pi) + s_4(t) \cos(\phi + \pi) \right)
+ \frac{1}{\pi} \sin \overline{\theta} \left( s_1(t) \sin \phi - s_2(t) \sin \phi + s_3(t) \sin(\phi + \pi) - s_4(t) \sin(\phi + \pi) \right)
\]

Thus, we get

\[
A_0(t) = \frac{(s_1(t) + s_2(t) + s_3(t) + s_4(t))}{2\pi}
\]

(5)

\[
A_1(t) = \frac{1}{\pi} \cos \phi \left( s_1(t) + s_2(t) - s_3(t) - s_4(t) \right)
\]

(6)

\[
B_1(t) = \frac{1}{\pi} \sin \phi \left( s_1(t) - s_2(t) - s_3(t) + s_4(t) \right)
\]

(7)

Ultimately, we can represent the above functions such that the voltage matrix \( \mathbf{V} = (v_1 \ v_2 \ v_3)^T \) can be represented as

\[
\mathbf{V} = \mathbf{MA}
\]

where \( \mathbf{M} \) is some pickup matrix and \( \mathbf{A} = (A_0(t) \ A_1(t) \ B_1(t))^T \). Additionally, from the equations above, we can represent \( \mathbf{A} \) as \( \mathbf{A} = \mathbf{XS} \) where \( \mathbf{S} = (s_1(t) \ s_2(t) \ s_3(t) \ s_4(t))^T \) and

\[
\mathbf{X} = \frac{1}{2\pi} \begin{bmatrix}
1 & 1 & 1 & 1 \\
2 \cos \phi & 2 \cos \phi & -2 \cos \phi & -2 \cos \phi \\
2 \sin \phi & -2 \sin \phi & -2 \sin \phi & 2 \sin \phi 
\end{bmatrix}
\]

is the \( 4 \times 3 \) mixing matrix. Then we can attempt to convert \( \mathbf{X} \) to row echelon form. We represent the equation \( \mathbf{A} = \mathbf{XS} \) in augmented form, such that

\[
\begin{bmatrix}
1 & 1 & 1 & 1 & 2\pi A_0(t) \\
2 \cos \phi & 2 \cos \phi & -2 \cos \phi & -2 \cos \phi & 2\pi A_1(t) \\
2 \sin \phi & -2 \sin \phi & -2 \sin \phi & 2 \sin \phi & 2\pi B_1(t)
\end{bmatrix}
\]
Multiplying the second row by \( \cos \phi \) and the third row by \( \sin \phi \), and replacing the second row by row 2 + row 3 and replacing the third row by row 2 - row 3, we get
\[
\begin{bmatrix}
1 & 1 & 1 & 1 & 2\pi A_0(t) \\
\cos 2\phi & 2 & -2 \cos 2\phi & -2 & 2\pi (\cos \phi A_1(t) + \sin \phi B_2(t)) \\
2 \cos 2\phi & 2 & -2 \cos 2\phi & -2 & 2\pi (\cos \phi A_1(t) - \sin \phi B_1(t))
\end{bmatrix}
\]
Further subtracting two times the first row from the second and \( 2 \cos 2\phi \) times the first row from the third, we reach
\[
\begin{bmatrix}
1 & 1 & 1 & 1 & 2\pi A_0(t) \\
\cos 2\phi & 2 & -2 \cos 2\phi & -2 & 2\pi (\cos \phi A_1(t) + \sin \phi B_2(t) - 2A_0(t)) \\
0 & -2(\cos 2\phi - 1) & -4 \cos 2\phi & -2(\cos 2\phi + 1) & 2\pi (\cos \phi A_1(t) - \sin \phi B_1(t) - 2\cos 2\phi A_0(t))
\end{bmatrix}
\]
Lastly, replacing the third row with the row 2 + row 3, we reach the desired result of
\[
\begin{bmatrix}
1 & 1 & 1 & 1 & 2\pi A_0(t) \\
\cos 2\phi & 2 & -2 \cos 2\phi & -2 & \pi (\cos \phi A_1(t) + \sin \phi B_2(t) - 2A_0(t)) \\
0 & 0 & - \cos 2\phi & -2(\cos 2\phi + 1) & \pi (\cos \phi A_1(t) - A_0(t)(\cos 2\phi + 1))
\end{bmatrix}
\]
Moreover, we express the pickup matrix \( \mathbf{M} \) as
\[
\begin{bmatrix}
2\pi a_0 & \pi a_1 \cos \theta & \pi a_1 \sin \theta \\
2\pi a_0 & \pi a_1 \cos \theta & \pi a_1 \sin \theta \\
2\pi a_0 & \pi a_1 \cos \theta & \pi a_1 \sin \theta
\end{bmatrix}
\]
Since we chose \( \theta = 0, 2\pi/3, 4\pi/3 \), we can simplify the above matrix to
\[
\begin{bmatrix}
2\pi a_0 & \pi a_1 & 0 \\
2\pi a_0 & -\pi a_1 & \sqrt{3}\pi a_1 \\
2\pi a_0 & -\pi a_1 & -\sqrt{3}\pi a_1
\end{bmatrix}
\]
Ultimately if we could find an invertible system of equations to relate the sound field to the reproduced speaker sound field, our frequency independent case will be complete. To start off, we note that since the each voltage function is an input for each speaker, there must be some system of equations relating \( \mathbf{S} \) and \( \mathbf{V} \). For simplicity, we denote this by the \( 4 \times 3 \) output matrix \( \Sigma \), such that \( \mathbf{S} = \Sigma \mathbf{V} \). From our previous equation, we can expand this to \( \mathbf{S} = \Sigma M \mathbf{A} \) and rewrite the original equation \( \mathbf{A} = \mathbf{X} \mathbf{S} \) as \( \mathbf{A} = \mathbf{X} \Sigma M \mathbf{A} \). If we define a \( 4 \times 3 \) ‘conversion’ matrix \( \Sigma \) to be \( \Sigma = \Sigma \mathbf{M} \), then we can see that \( \Sigma \) must be the right inverse of \( \mathbf{X} \), such that \( \mathbf{X} \Sigma = \mathbf{I} \) where \( \mathbf{I} \) is the \( 3 \times 3 \) identity matrix. Then
\[
\frac{1}{2\pi} \begin{bmatrix}
1 & 1 & 1 & 1 \\
\cos \phi & \cos \phi & -2 \cos \phi & -2 \cos \phi \\
\sin \phi & -2 \sin \phi & -2 \sin \phi & 2 \sin \phi
\end{bmatrix}
\begin{bmatrix}
\sigma_{11} & \sigma_{12} & \sigma_{13} \\
\sigma_{21} & \sigma_{22} & \sigma_{23} \\
\sigma_{31} & \sigma_{32} & \sigma_{33} \\
\sigma_{41} & \sigma_{42} & \sigma_{43}
\end{bmatrix}
= \begin{bmatrix} 1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \end{bmatrix}
\]
Simplifying \( \Sigma \) to satisfy the above equation, we see that
\[
\Sigma = \frac{\pi}{2} \begin{bmatrix}
1 & \sec \phi & \csc \phi \\
\frac{\sec \phi}{2} & -\frac{\csc \phi}{2} \\
\frac{1}{2} & -\frac{\sec \phi}{2} & -\frac{\csc \phi}{2} \\
\frac{1}{2} & \frac{\sec \phi}{2} & \frac{\csc \phi}{2}
\end{bmatrix}
\]
Hence, we reach a right-invertible linear transformation to convert the original sound field function to the speaker function, where \( \mathbf{A} = \mathbf{X} \mathbf{S}, \Sigma = \Sigma \mathbf{A}, \) and \( \mathbf{X} \Sigma = \mathbf{I} \), as desired.
3 Frequency Dependent Case

In reality, the pickup function of microphones are dependent on the frequency of the sound field, such that

$$\mu(\omega; \theta; \bar{\theta}) = a_0(\omega) + a_1(\omega)\cos(\theta - \bar{\theta})$$

The trouble arises when we try to determine how to reach \(v(t)\) from a pickup function dependent on frequency and a sound field function of time. We can either compute the sound field function as a function of frequency, \(\hat{f}(\omega, \theta)\), and further compute the voltage as a function of frequency, such that

$$v(\omega; \bar{\theta}) = \int_{-\pi}^{\pi} \mu(\omega; \theta; \bar{\theta}) \hat{f}(\omega, \theta) d\theta$$

$$= \int_{-\pi}^{\pi} \int_{-\infty}^{\infty} \mu(\omega; \theta; \bar{\theta}) f(s, \theta) e^{-i\omega s} ds d\theta$$

and hence take the inverse Fourier transform to find

$$v(t; \bar{\theta}) = \int_{-\infty}^{\infty} \int_{-\pi}^{\pi} \int_{-\infty}^{\infty} \mu(\omega, \theta; \bar{\theta}) f(s, \theta) e^{i\omega(t-s)} dtd\theta ds$$

(9)

Or, we could alternatively transform the pickup function \(\mu(\omega, \theta; \bar{\theta})\) into the time domain such that

$$\hat{\mu}(t, \theta; \bar{\theta}) = \int_{-\infty}^{\infty} \mu(\omega, \theta; \bar{\theta}) e^{i\omega t} d\omega$$

Then we can apply the original equation (1), such that

$$v(t; \bar{\theta}) = \int_{-\pi}^{\pi} \int_{-\infty}^{\infty} \mu(\omega, \theta; \bar{\theta}) f(t, \theta) d\omega d\theta$$

However, we make the following claim to assert the two are not equal. In either case, we expect a solution that physically makes sense.

**Proposition** The two representations of the voltage function are not equal. In other words,

$$\int_{-\infty}^{\infty} \int_{-\pi}^{\pi} \int_{-\infty}^{\infty} \mu(\omega, \theta; \bar{\theta}) f(s, \theta) e^{i\omega(t-s)} ds d\theta d\omega \neq \int_{-\pi}^{\pi} \int_{-\infty}^{\infty} \mu(\omega, \theta; \bar{\theta}) f(t, \theta) e^{i\omega t} d\omega d\theta$$

**Proof** Suppose, to the contrary, that the two representations are equal. For simplicity, let us denote the Fourier transform as \(F\) and the inverse Fourier Transform as \(F^{-1}\). Then

$$\int_{-\infty}^{\infty} \int_{-\pi}^{\pi} \int_{-\infty}^{\infty} \mu(\omega, \theta; \bar{\theta}) f(s, \theta) e^{i\omega(t-s)} ds d\theta d\omega = \int_{-\pi}^{\pi} \int_{-\infty}^{\infty} \mu(\omega, \theta; \bar{\theta}) f(t, \theta) e^{i\omega t} d\omega d\theta$$

$$= \int_{-\pi}^{\pi} \left\{ \mu(\omega, \theta) \right\} * \left\{ F^{-1} \{ f(t, \theta) \} \right\} d\theta$$

$$= \int_{-\pi}^{\pi} \left\{ \mu(\omega, \theta) \right\} * f(t, \theta) d\theta$$

Where \(*\) denotes convolution. Alternatively,

$$\int_{-\pi}^{\pi} \int_{-\infty}^{\infty} \mu(\omega, \theta; \bar{\theta}) f(t, \theta) e^{i\omega t} d\omega d\theta = \int_{-\pi}^{\pi} \left\{ \mu(\omega, \theta; \bar{\theta}) \right\} f(t, \theta) d\theta$$
Since convolution and scalar multiplication are not identical binary operators, it follows that the two equations are not equal.

QED

From henceforth we will use the first triple integral to denote the voltage of a frequency dependent microphone. Though we do not provide a physical proof of why this is the correct equation in this text, it follows from the fact that the second equation provides inaccurate results for a frequency dependent voltage pickup pattern. Using the notation from Section 2 above, it follows that

\[ v(t, \theta) = \int_{-\infty}^{\infty} \int_{-\pi}^{\pi} \int_{-\infty}^{\infty} \left( a_0(\omega) + a_1(\omega) \cos(\theta - \theta) \right) A_0(s) + A_1(s) \cos \theta + B_1(s) \sin \theta e^{i\omega(t-s)} ds d\theta d\omega \]

\[ = 2\pi \int_{-\infty}^{\infty} \int_{-\pi}^{\pi} \int_{-\infty}^{\infty} a_0(\omega) A_0(s) e^{i\omega(t-s)} ds d\omega + \pi \cos \theta \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} a_1(\omega) A_1(s) e^{i\omega(t-s)} ds d\omega \]

\[ + \pi \sin \theta \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} a_1(\omega) B_1(s) e^{i\omega(t-s)} ds d\omega \]

\[ = 2\pi \mathcal{F}^{-1} \{ a_0(\omega) \mathcal{F} \{ A_0(s) \} \} + \pi \cos \theta \mathcal{F}^{-1} \{ a_1(\omega) \mathcal{F} \{ A_1(s) \} \} + \pi \sin \theta \mathcal{F}^{-1} \{ a_1(\omega) \mathcal{F} \{ B_1(s) \} \} \]

serves as a solution to a first degree microphone pickup function. Since we still expect the speaker sound field function to be solely dependent on time, the function

\[ v_{\text{speaker}}(t, \theta) = \int_{-\infty}^{\infty} \int_{-\pi}^{\pi} \int_{-\infty}^{\infty} (a_0(\omega) + a_1(\omega) \cos(\theta - \theta)) (s_1(t) \delta(\theta - \phi) + s_2(t) \delta(\theta + \phi) + s_3(t) \delta(\theta - \phi - \pi) + s_4(t) \delta(\theta + \phi + \pi)) e^{i\omega(t-s)} ds d\theta d\omega \]

\[ = \mathcal{F}^{-1} \{ a_0(\omega) \mathcal{F} \{ s_1(t) + s_2(t) + s_3(t) + s_4(t) \} \} + \mathcal{F}^{-1} \{ a_1(\omega) \mathcal{F} \{ s_1(t) \cos(\phi - \theta) + s_2(t) \cos(\phi + \theta) + s_3(t) \cos(\phi - \theta) + s_4(t) \cos(\phi + \pi + \theta) \} \} \]

still holds. By requiring that \( v_{\text{speaker}}(t, \theta) = v(t, \theta) \) as in the frequency independent case, we get a similar equality to section 2 above, aside from the forward and reverse Fourier transforms.

\[ 2\pi \mathcal{F}^{-1} \{ a_0(\omega) \mathcal{F} \{ A_0(s) \} \} = \mathcal{F}^{-1} \{ a_0(\omega) \mathcal{F} \{ s_1(s) + s_2(s) + s_3(s) + s_4(s) \} \} \]

\[ \pi \cos \theta \mathcal{F}^{-1} \{ a_1(\omega) \mathcal{F} \{ A_1(s) \} \} = \cos \theta \mathcal{F} \mathcal{F}^{-1} \{ a_1(\omega) \mathcal{F} \{ s_1(s) + s_2(s) - s_3(s) - s_4(s) \} \} \]

\[ \pi \sin \theta \mathcal{F}^{-1} \{ a_1(\omega) \mathcal{F} \{ B_1(s) \} \} = \sin \theta \sin \phi \mathcal{F} \mathcal{F}^{-1} \{ a_1(\omega) \mathcal{F} \{ s_1(s) + s_2(s) - s_3(s) + s_4(s) \} \} \]

Since \( \cos \phi, \sin \phi, \text{ and } \cos \theta \) are constants we are able to take the inverse operators of the reverse and forward transforms, respectively. If we assume \( a_0(\omega) \) and \( a_1(\omega) \) are nonzero, then we are able to cancel out \( a_0(\omega) \) and \( a_1(\omega) \) from both equations, such that we are left with the original solution from section 2 above:

\[ A_0(t) = \frac{(s_1(t) + s_2(t) + s_3(t) + s_4(t))}{2\pi} \]

\[ A_1(t) = \frac{1}{\pi} \cos \phi \left( s_1(t) + s_2(t) - s_3(t) - s_4(t) \right) \]

\[ B_1(t) = \frac{1}{\pi} \sin \phi \left( s_1(t) - s_2(t) - s_3(t) + s_4(t) \right) \]

Thus, the solution to the ambisonic system is independent of frequency, regardless of the dependency of the microphone. This is primarily due to the basic fact that the Fourier transform is a linear operator,
and the distribution between the sound field function and the speaker function do not change depending on frequency.

When we consider the original system of microphones directed at $\theta = 0, \frac{2\pi}{3}, \frac{4\pi}{3}$, we can represent this solution as $V(t) = \mathcal{F}^{-1}\{M(\omega)F\{A(t)\}\}$. Since the speaker matrix is not affected by frequency or time, as it serves to convert the voltage function to the speaker sound function. Thus, we get $S(t) = \Sigma V(t) = \Sigma \mathcal{F}^{-1}\{M(\omega)F\{A(t)\}\}$. However, if we wish to represent the $4 \times 3$ ‘conversion’ matrix as a function of time, we must transform the microphone pickup matrix from the frequency domain to the time domain. Thus, $\Sigma(t) = \Sigma \mathcal{F}^{-1}\{M(\omega)\}$ and $S(t) = \Sigma(t)A(t) = \Sigma \mathcal{F}^{-1}\{M(\omega)\}A(t)$. Since $X$ is the right inverse of $\Sigma(t)$, $X$ must also be a function of time. From the equation above, we know

$$A(t) = X(t)S(t) = X(t)\Sigma \mathcal{F}^{-1}\{M(\omega)\}A(t)$$

(10)

Hence, there still exists an invertible mapping between the original sound field function and the reproduced speaker function. Since the system of solutions for the sound field functions in terms of the four speaker functions is unchanged, it directly follows that the matrices will be unchanged in the frequency independent solution.

### 4 Conclusion

The original question raised was whether a microphone array of 3 unidirectional microphones surrounded by an array of 4 speakers would be ambisonic. We divided the equation into two cases: one where the microphones were independent of frequency, and one where the pickup functions were functions of both time and direction. From the calculations in sections 2 and 3 above, we see that the solution to both cases is indeed the same. Namely, $S = \Sigma A$ where

$$\Sigma = \frac{\pi}{2} \begin{bmatrix} 1 & \sec \phi & \csc \phi \\ 1 & \frac{2}{\sec \phi} & \frac{2}{\csc \phi} \\ 1 & \frac{2}{\sec \phi} & \frac{2}{\csc \phi} \\ 1 & \frac{2}{\sec \phi} & \frac{2}{\csc \phi} \end{bmatrix}$$

$$S = \begin{bmatrix} s_1(t) \\ s_2(t) \\ s_3(t) \\ s_4(t) \end{bmatrix}$$

$$A = \begin{bmatrix} A_0(t) \\ A_1(t) \\ B_1(t) \end{bmatrix}$$

From this solution, and noting that this matrix is invertible to also map from $S$ to $A$, we can see that the system is indeed ambisonic. Thus, any sound field can be reproduced through a first degree microphone (i.e. its respective Fourier expansion only includes the first order terms) regardless of its dependency on frequency.